Solutions to H.W #6

1. Let $f, g \in \mathcal{R}_\alpha[a,b]$ be two functions that satisfy $f \leq g$. Then, for any partition $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a,b]$, $m_{g_i} = \inf_{x_{i-1} \leq x \leq x_i} g(x) \geq m_{f_i} = \inf_{x_{i-1} \leq x \leq x_i} f(x)$ for all $i \in \{1, \ldots, n\}$.

Thus $L(f,P) = \sum_{i=1}^{n} m_{f_i} \Delta\alpha_i \leq \sum_{i=1}^{n} m_{g_i} \Delta\alpha_i = L(g,P)$

and it follows that

$$\int_a^b f\,d\alpha = \underline{\int_a}^b f\,d\alpha = \sup_P L(f,P) \leq \sup_P L(g,P) =$$

$$= \underline{\int_a}^b g\,d\alpha = \int_a^b g\,d\alpha.$$

2. Suppose $f, g \in R_\alpha[a,b]$. Then, for any partition P of $[a,b]$, $U(f+g, P) \leq U(f,P) + U(g,P)$ and $L(f+g,P) \geq L(f,P) + L(g,P)$ (why?)

Thus $\overline{\int_a^b} (f+g)\,d\alpha = \inf_P U(f+g, P) \leq \inf_P U(f,P) + \inf_P U(g,P)$
$= \overline{\int_a^b} f\,d\alpha + \overline{\int_a^b} g\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$ and

$$\underline{\int_a}^b (f+g)\,d\alpha \geq \underline{\int_a}^b f\,d\alpha + \underline{\int_a}^b g\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$$

Hence

$$\int_a^b f\,d\alpha + \int_a^b g\,d\alpha \leq \underline{\int_a^b}(f+g)\,d\alpha \leq \overline{\int_a^b}(f+g)\,d\alpha \leq \int_a^b f\,d\alpha +$$
$$+ \int_a^b g\,d\alpha$$

which implies that $\underline{\int_a^b}(f+g)\,d\alpha = \overline{\int_a^b}(f+g)\,d\alpha$ (i.e. $f+g \in R_\alpha[a,b]$)

and $\int_a^b (f+g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$.

3. Suppose $f \in R_\alpha[a,b]$. Then, for any $x, y \in [a,b]$

$\big|\,|f(x)| - |f(y)|\,\big| \leq |f(x) - f(y)|$. Thus, for any $[c,d] \subset [a,b]$

$$\sup_{c\leq x\leq d}|f(x)| - \inf_{c\leq x\leq d}|f(x)| \leq \sup_{c\leq x\leq d} f(x) - \inf_{c\leq x\leq d} f(x).$$

Let P be a partition of $[a,b]$ for which $U(f,P) - L(f,P) < \epsilon$

Then $U(|f|,P) - L(|f|,P) \leq U(f,P) - L(f,P) < \epsilon$

which implies that $|f| \in R_\alpha[a,b]$.

Finally, observe that $|f| \geq \max\{f, -f\}$. Therefore, by problem 1, $\int_a^b |f|\,d\alpha \geq \int_a^b \max\{f,-f\}\,d\alpha \geq \pm\int_a^b f\,d\alpha$.

In particular, $\int_a^b |f|\,d\alpha \geq \left|\int_a^b f\,d\alpha\right|$.

4. Assume $f \in R_\alpha[a,b]$. We will show that $f^2 \in R_\alpha[a,b]$.

If $[c,d] \subset [a,b]$, Observe that $M_{f^2} = \sup_{c\leq x\leq d} f^2(x) =$

$= \left(\sup_{c\leq x\leq d} f(x)\right)^2 = (M_f)^2$. Similarly, $m_{f^2} = \inf_{c\leq x\leq d} f^2(x) = (m_f)^2$.

Also, notice that $f^2(x) - f^2(y) = (f(x)+f(y))(f(x)-f(y)) \leq$
$\leq 2\|f\|_\infty (f(x)-f(y))$.

Let P be a partition of $[a,b]$ for which

$$\mathcal{U}(f,P) - \mathcal{L}(f,P) < \frac{\epsilon}{2\|f\|_\infty}$$

Then $\mathcal{U}(f^2,P) - \mathcal{L}(f^2,P) = \sum_{i=1}^{n} (M_{f^2 i} - m_{f^2 i}) \Delta\alpha_i$

$$= \sum_{i=1}^{n} (M_{fi}^2 - m_{fi}^2)\Delta\alpha_i = \sum_{i=1}^{n} (M_{fi}+m_{fi})(M_{fi}-m_{fi})\Delta\alpha_i$$

$$\leq 2\|f\|_\infty \sum_{i=1}^{n} (M_{fi}-m_{fi})\Delta\alpha_i = 2\|f\|_\infty \left(\mathcal{U}(f,P)-\mathcal{L}(f,P)\right)$$

$$< 2\|f\|_\infty \frac{\epsilon}{2\|f\|_\infty} = \epsilon.$$

Let $f,g \in R_\alpha[a,b]$. Then $fg = \dfrac{(f+g)^2-(f-g)^2}{4}$

implies, by problem 2 and the above result, that
$fg \in R_\alpha[a,b]$.

5. Suppose $f \in R_\alpha[a,b]$ and $[c,d] \subset [a,b]$. Then there is a partition P of $[a,b]$ that contains the points c and d such that

$$\mathcal{U}(f,P) - \mathcal{L}(f,P) < \epsilon.$$

Let $Q = P \cap [c,d]$ be a partition of $[c,d]$. Then

$$U(f,Q) - L(f,Q) < U(f,P) - L(f,P) < \epsilon$$

implying that $f \in R_\alpha[c,d]$.

To show that $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$ for any $a<c<b$, let P be any partition of $[a,b]$ that contains c. Set $Q = [a,c]\cap P$ and $T = [c,d]\cap P$. Then, if $P^* \supset P$, $Q^* \supset Q$, and $T^* \supset T$,

$$L(f,P) = L(f,Q) + L(f,T) \le L(f,Q^*) + L(f,T^*) \le$$

$$\le \underline{\int_a^c} f\,d\alpha + \underline{\int_c^b} f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

Thus $\int_a^b f\,d\alpha = \underline{\int_a^b} f\,d\alpha = \sup_P L(f,P) \le \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$.

Similarly, $\overline{\int_a^b} f\,d\alpha \ge \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$, because

$$U(f,P) \ge U(f,P^*) = U(f,P^*\cap[a,c]) + U(f,P^*\cap[c,b])$$

$$\ge \overline{\int_a^c} f\,d\alpha + \overline{\int_c^b} f\,d\alpha.$$

Hence $\int_a^c f\,d\alpha + \int_c^b f\,d\alpha \le \int_a^b f\,d\alpha \le \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$,

which shows the desired result.

6. Suppose $f \in R_\alpha[a,b]$. For $a \le x \le b$, define $F(x) = \int_a^x f\,d\alpha$.

Let $P = \{a_0 < x_1 < x_2 < \ldots < x_n = b\}$ be a partition of $[a,b]$.

Then $V(F,P) = \sum_{i=1}^n |F(x_i) - F(x_{i-1})| = \sum_{i=1}^n \left|\int_a^{x_i} f\,d\alpha - \int_a^{x_{i-1}} f\,d\alpha\right|$

$$= \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f\,d\alpha \right| \leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f|\,d\alpha = \int_a^b |f|\,d\alpha = K < \infty$$

since $|f| \in R_\alpha[a,b]$

Hence $V_a^b F = \sup_P V(F,P) \leq K$, implying that $F \in BV[a,b]$.

Furtheremore, if $\alpha \in C[a,b]$, then α is uniformly continuous. In particular, given $\epsilon > 0$, there is a $\delta > 0$ s.t. $\alpha(y) - \alpha(x) < \epsilon$ whenever $x < y < x + \delta$. Then

$$|F(y) - F(x)| = \left| \int_x^y f\,d\alpha \right| \leq \int_x^y \|f\|_\infty\,d\alpha = \|f\|_\infty (\alpha(y) - \alpha(x))$$

$$< \|f\|_\infty \epsilon$$

implying that F is continuous.

7. Suppose $\int_a^b f\,d\alpha = 0$ for every $f \in C[a,b]$. Then, in particular, $\int_a^b d\alpha = \alpha(b) - \alpha(a) = 0$, since $1 \in C[a,b]$. By hypothesis, α is increasing. Hence, α is not a constant iff $\alpha(b) - \alpha(a) > 0$. Thus, α is constant iff $\alpha(b) - \alpha(a) = 0$, which proves the claim of problem 7.

8. Suppose $f \in R_\alpha[a,b]$ and $U(f,P) - L(f,P) < \epsilon$ for some partition P. Then

$$L(f,P) \leq \underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha \leq U(f,P)$$

and

$$L(f,P) \leq \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i \leq U(f,P)$$

In other words,

$$\sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i\,, \quad \underline{\int_a^b} f\,d\alpha = \int_a^b f\,d\alpha \in \big(L(f,P),\, U(f,P)\big)$$

so $$\left| \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - \int_a^b f\,d\alpha \right| \leq \ell\Big(L(f,P),\, U(f,P)\Big) =$$

$$= U(f,P) - L(f,P) < \epsilon .$$

9. Let $\epsilon > 0$. By hypothesis, there is some partition P of $[a,b]$ such that $\left| \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - I \right| < \epsilon$ for any choice of points $t_i \in [x_{i-1}, x_i]$.

Then we may take t_i for which $M_i - \frac{\epsilon}{\alpha(b)-\alpha(a)} < f(t_i)$ which yields $|U(f,P) - I| = \left| \sum_{i=1}^{n} M_i\,\Delta\alpha_i - I \right| =$

$$= \left| \sum_{i=1}^{n} \left(M_i - \frac{\epsilon}{\alpha(b)-\alpha(a)}\right)\Delta\alpha_i - I + \sum_{i=1}^{n} \frac{\epsilon}{\alpha(b)-\alpha(a)}\,\Delta\alpha_i \right| \leq$$

$$\leq \sum_{i=1}^{n} f(t_i)\,\Delta\alpha_i - I + \epsilon < 2\epsilon$$

A similar calculation shows that $|I - L(f,P)| < 2\epsilon$.

Thus $U(f,P) - L(f,P) < 4\epsilon$, which shows that $f \in R_\alpha[a,b]$

It is easy to see that for any $\epsilon > 0$, there is a partition P such that

$$L(f, P),\ U(f, P) \in (I - \epsilon, I + \epsilon).$$ Thus $\int_a^b f\,d\alpha \in (I - \epsilon, I + \epsilon)$.

This proves that $I = \int_a^b f\,d\alpha$.

10. Assume $U(f, P) - L(f, P) < \epsilon$. Then

$$\sum_{i=1}^{n} |f(t_i) - f(s_i)|\,\Delta\alpha_i \leq \sum_{i=1}^{n} \sup_{t_i, s_i \in [x_{i-1}, x_i]} |f(t_i) - f(s_i)|\,\Delta\alpha_i =$$

$$= \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i = U(f, P) - L(f, P) < \epsilon.$$

11. Suppose f and α share a common-sided discontinuity at some point $a < c < b$. Without loss of generality, $f(c+) \neq f(c)$ and $\alpha(c+) \neq \alpha(c)$. Let P be any partition that contains the point c. Say $c = x_K$.

Then $$U(f, P) - L(f, P) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta\alpha_i \geq (M_{K+1} - m_{K+1})\,\Delta\alpha_{K+1}$$

$$= \sup_{t_{K+1}, s_{K+1} \in [x_K, x_{K+1}]} |f(t_{K+1}) - f(s_{K+1})|\,\Delta\alpha_{K+1} \geq \lim_{\delta \to 0^+} \omega(f, [c, c+\delta])\,\omega(\alpha, [c, c+\delta])$$

$$= \omega_f(c)\,\omega_\alpha(c) > 0$$

Thus, if we set $\epsilon = \omega_f(c)\,\omega_\alpha(c)$, we see that

$\mathcal{U}(f, P^*) - \mathcal{L}(f, P^*) \geq \epsilon$ for any refinement $P^* \supset P$.
Consequently $f \notin R_\alpha[a,b]$.

12. Suppose $f \in B[a,b]$ is not continuous at some $a<c<b$. Without loss of generality, $f(c+) \neq f(c)$. Let $\alpha = \chi_{(c,b]}$. Then $1 = \alpha(c+) \neq 0 = \alpha(c)$.
Hence, by problem 11, $f \notin R_\alpha[a,b]$. We have shown that a function f that is not everywhere continuous on $[a,b]$ cannot belong to $R_\alpha[a,b]$ for all increasing α. Thus $\bigcap\{R_\alpha[a,b] : \alpha \text{ increasing}\} \subseteq C[a,b]$. Conversely, $C[a,b] \subseteq R_\alpha[a,b]$ for any α. (Why?)

13. Suppose that α is continuous and $f, g \in R_\alpha[a,b]$ differ at finitely many points $c_1 < c_2 < \dots < c_n \in [a,b]$.
We will show that $\int_a^b (g-f)\,d\alpha = 0$. Since α is actually uniformly continuous, we may pick for each $\epsilon > 0$ a $\delta > 0$ such that $c_j \notin [c_i - \delta, c_i + \delta]$ for any $i \neq j \in \{1, \dots, n\}$ and $|\alpha(x) - \alpha(y)| < \epsilon$ whenever $|x-y| \leq 2\delta$.
Then $\left|\int_a^b (g-f)\,d\alpha\right| = \left|\sum_{i=1}^n \int_{c_i-\delta}^{c_i+\delta} (g-f)\,d\alpha\right| \leq \sum_{i=1}^n \int_{c_i-\delta}^{c_i+\delta} \|g-f\|_\infty\,d\alpha$
$= \|g-f\|_\infty \sum_{i=1}^n [\alpha(c_i+\delta) - \alpha(c_i-\delta)] \leq n\|g-f\|_\infty \epsilon$. This implies that $\left|\int_a^b (g-f)\,d\alpha\right|$ is smaller than any positive number, which

proves our claim.

Now suppose that $f, g \in R_\alpha[a,b]$ differ on countably many values. We will prove that $\int_a^b f\,d\alpha = \int_a^b g\,d\alpha$.

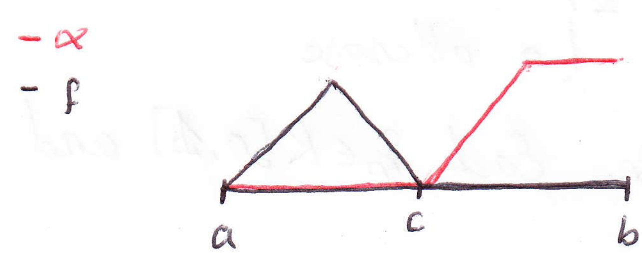
Observe that $\left|\int_a^b (g-f)\,d\alpha\right| \leq \int_a^b |g-f|\,d\alpha$, where $|g-f| \in R_\alpha[a,b]$. Clearly, $L(|g-f|, P) = 0$ for any partition P of $[a,b]$. Thus

$$\overline{\int_a^b} |g-f|\,d\alpha = \underline{\int_a^b} |g-f|\,d\alpha = 0,$$

which implies that $\int_a^b (g-f)\,d\alpha = 0$.

Remark: Observe that if $f = \chi_{\mathbb{Q}}$ and $g = 0$, then f and g differ on countably many values. Yet $\overline{\int_a^b} f\,dx \neq \overline{\int_a^b} g\,dx$. This is not in violation of the above statement, however, since $f \notin R_x[a,b]$.

14. Let α and f be two functions that correspond to the picture below



Clearly, $f, \alpha \in C[a,b]$ and $\int_a^b |f|\,d\alpha = 0$ even though

$|f|$ is nonzero.

15. Suppose $f \in C[a,b]$ and $f(x_0) \neq 0$ for some x_0. Then $|f(x_0)| > 0$. This implies that $|f|$ is nonzero is some subinterval $[c,d]$ that contains x_0.
Let P be any partition of $[a,b]$ that contains c and d.
Then $\underline{\int_a^b} |f| dx \geq L(|f|, P) \geq \min_{x \in [c,d]} |f(x)| (d-c) > 0.$

Thus, $\|f\| = \int_a^b |f| dx$ defines a norm on $C[a,b]$.
Unfortunately, $\|\cdot\|$ does not define a norm on $R[a,b]$:
Let

$$f(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \in (a,b] \end{cases}$$

Then $\int_a^b |f| dx = 0$ even though $|f|$ is a nonzero function.

16. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of all the rationals in $[0,1]$. Define $f_n : [0,1] \mapsto \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \{r_1, \dots, r_n\} \\ 0 & \text{otherwise} \end{cases}$$

Then $f_n \mapsto \chi_{\mathbb{Q}}$ pointwise. Each $f_n \in R[0,1]$ and $\int_0^1 f_n dx = 0$.
$\chi_{\mathbb{Q}} \notin R[0,1]$.